

# The Plancherel Formula for the Universal Covering Group of $SL(2, R)$ Revisited

Debabrata Basu

*Saha Institute of Nuclear Physics, 1/AF, Bidhannagar, Kolkata - 700064, India\**

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The Plancherel formula for the universal covering group of  $SL(2, R)$  derived earlier by Pukanszky on which Herb and Wolf build their Plancherel theorem for general semisimple groups is reconsidered. It is shown that a set of unitarily equivalent representations is treated by these authors as distinct. Identification of this equivalence results in a Plancherel measure  $(s \operatorname{Re} \tanh \pi(s + \frac{i\tau}{2}), 0 \leq \tau < 1)$  which is different from the Pukanszky-Herb-Wolf measure  $(s \operatorname{Re} \tanh \pi(s + i\tau), 0 \leq \tau < 1)$ .

## I. INTRODUCTION

A major step ahead of the traditional concept of character was taken by Gel'fand and Naimark[1] in their definition in terms of the integral kernel of the group ring which defines character as a linear functional on the group manifold. Let us denote by  $X$  the set of infinitely differentiable functions  $x(g)$  on the group, which are equal to zero outside a bounded set. If  $g \rightarrow T_g$  is a representation of the group  $G$  we define the operator of the group ring as

$$T_x = \int d\mu(g) x(g) T_g \quad (1)$$

where  $d\mu(g)$  is the left and right invariant measure (assumed coincident) on  $G$  and the integration extends over the entire group manifold. If we define

$$x_1 x_2(g) = x(g) = \int x_1(g_1) x_2(g_1^{-1} g) d\mu(g_1)$$

then

$$T_{x_1 x_2} = T_{x_1} T_{x_2} \quad (2)$$

Let us suppose that  $g \rightarrow T_g$  is a unitary representation of the group  $G$  realized in the Hilbert space  $H$  of the functions  $f(z)$  with the scalar product

$$(f, g) = \int \overline{f(z)} g(z) d\lambda(z) \quad (3)$$

where  $d\lambda(z)$  is the measure in  $H$ . Then the operator  $T_x$  is an integral operator with a kernel

$$T_x f(z) = \int K(z, z_1) f(z_1) d\lambda(z_1) \quad (4)$$

It then follows that  $K(z, z_1)$  is a positive definite Hilbert-Schmidt kernel satisfying

$$\int |K(z, z_1)|^2 d\lambda(z) d\lambda(z_1) < \infty$$

Such a kernel has a trace

$$\operatorname{Tr}(T_x) = \int K(z, z) d\lambda(z) \quad (5)$$

Using the definition of the group ring  $\operatorname{Tr}(T_x)$  can be written in the form,

$$\operatorname{Tr}(T_x) = \int x(g) \pi(g) d\mu(g) \quad (6)$$

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\*Guest Scientist

The function  $\pi(g)$  is the character of the representation  $g \rightarrow T_g$ . It should be noted that in this definition the matrix representation of the group does not appear and it makes a complete synthesis of the finite and infinite dimensional irreducible unitary representations.

The Vilenkin-Klimyk[2] invariant method for the derivation of the Plancherel formula consists in inverting the integral transform (6) to get  $x(e)$ . An important step to achieve this objective is taken by evaluating the residue of the generalized function  $(x_3^2 - x_2^2 - x_1^2)_+^\lambda$  at the simple pole at  $\lambda = -\frac{3}{2}$ .

Our present framework is, however, more general than that of Vilenkin and Klimyk insofar as it is not restricted to the integral and half integral representations of Bargmann[4].

The irreducible unitary representations of the universal covering group of  $SL(2, R)$  (denoted by  $\widetilde{SL}(2, R)$ ) consists of four types:

(a) Principal series of representations  $C_s^\epsilon$ :

$$J_1^2 + J_2^2 - J_3^2 = k(1 - k); \quad k = \frac{1}{2} + is$$

$$m = \epsilon \pm n, \quad n = 0, 1, 2, \dots; \quad 0 \leq \epsilon < 1$$

(b) Positive discrete series  $D_k^+$ :

$$k > 0; \quad m = k + n, \quad n = 0, 1, 2, \dots$$

(c) Negative discrete series  $D_k^-$ :

$$k > 0, \quad m = -k - n, \quad n = 0, 1, 2, \dots$$

(d) Exceptional series of representations  $C_q^\epsilon$ :

$$\epsilon(1 - \epsilon) < q < \frac{1}{4}; \quad 0 \leq \epsilon < 1$$

$$m = \epsilon \pm n, \quad n = 0, 1, 2, \dots$$

We omit the exceptional representations as they do not appear in the calculations.

The Plancherel formula for  $\widetilde{SL}(2, R)$  was written down by Pukanszky[5]. Following Pukanszky's work the more general problem of Plancherel theorem for semisimple groups was attempted by Herb and Wolf[6] and by Duflo and Vergne[7] who claim to be in agreement with Pukanszky. The edifice of Herb and Wolf's work on general semisimple groups is built on Pukanszky's work on  $\widetilde{SL}(2, R)$  ("We also need this special result to do the general case.")[6]

However Pukanszky-Herb-Wolf formula for  $\widetilde{SL}(2, R)$  suffers from a serious flaw. All these authors treat a set of unitarily equivalent representations as distinct. This flaw in Pukanszky's work has been carried into the later work of Herb and Wolf[6]. As the rectification of the flaw for the general problem is likely to be more involved we start by investigating the problem for  $\widetilde{SL}(2, R)$ .

We now assert that the representations  $(\epsilon, s)$  and  $(1 - \epsilon, s)$  belonging to the principal series of  $\widetilde{SL}(2, R)$  are unitarily equivalent. The finite element of the group for the representation  $(\epsilon, s)$  is given by (in terms of  $SU(1, 1)$  parameters),

$$T_u^{(\epsilon, s)} f(z) = (\overline{\beta}z + \alpha)^{-k-\epsilon} (\beta z + \overline{\alpha})^{-k+\epsilon} f\left(\frac{\alpha z + \overline{\beta}}{\beta z + \overline{\alpha}}\right) \quad (7)$$

$$k = \frac{1}{2} + is, \quad 0 \leq \epsilon < 1, \quad |z| = 1$$

The representations  $C_s^\epsilon$  and  $C_s^{1-\epsilon}$  are equivalent because a fundamental property of the group ring ensures that

$$\text{Tr}[T_x^{(1-\epsilon, s)}] = \text{Tr}[T_x^{\epsilon, s}] \quad (8)$$

It has been shown in Sec.IB that the integral kernel of the ring is given by

$$T_{x, \eta}^{(\epsilon, s)} g(\theta) = \int_0^{2\pi} K_{(\eta, \epsilon)}(\theta, \theta_1) g(\theta_1) d\theta_1 \quad (9)$$

where

$$f(z) = f(-ie^{i\theta}) = g(\theta) \quad (10)$$

is a single valued function and

$$K_{(\eta,\epsilon)}^s(\theta, \theta_1) = \frac{1}{4} \int x(\underline{\theta}^{-1} \underline{k} \underline{\theta}_1) e^{i(\theta-\theta_1)\epsilon} |k_{22}|^{-1-2is} d\mu_l(k) \cos 2\pi\eta\epsilon \quad (11)$$

where following the traditional procedure of classical analysis we have defined the value of the functions on the real axis as

$$f(x) = \frac{1}{2} [f(x+io) + f(x-io)] \quad (12)$$

Thus for  $C_s^{1-\epsilon}$  we have

$$K_{(\eta,1-\epsilon)}^s(\theta, \theta_1) = \frac{1}{4} \int x(\underline{\theta}^{-1} \underline{k} \underline{\theta}_1) e^{i(\theta-\theta_1)(1-\epsilon)} |k_{22}|^{-1-2is} d\mu_l(k) \cos 2\pi\eta\epsilon \quad (13)$$

Thus although the integral kernels differ the  $\epsilon$  dependent term from the exponential drops out from the traces and we have

$$\begin{aligned} \text{Tr} [T_{x,\eta}^{(\epsilon,s)}] &= \int K_{(\eta,\epsilon)}^s(\theta, \theta) d\theta \\ &= \int K_{(\eta,1-\epsilon)}(\theta, \theta) d\theta = \text{Tr} [T_{x,\eta}^{(1-\epsilon,s)}] \end{aligned}$$

We may, therefore, write

$$\begin{aligned} \text{Tr} [T_x^{(\epsilon,s)}] &= \theta(2\epsilon)\theta(1-2\epsilon) \text{Tr} [T_x^{(\epsilon,s)}] + \theta(2-2\epsilon)\theta(2\epsilon-1) \text{Tr} [T_x^{\epsilon,s}] \\ &= \theta(2\epsilon)\theta(1-2\epsilon) \text{Tr} [T_x^{(\epsilon,s)}] + \theta(2-2\epsilon)\theta(2\epsilon-1) \text{Tr} [T_x^{(1-\epsilon,s)}] \end{aligned} \quad (14)$$

This point has not been taken care of by Pukanszky or by Herb and Wolf.

The problem consists of two parts (1) evaluation of the character of the representations of type (a), (b) and (c); (2) the inversion problem and the subsequent computation of  $x(e)$ . We now announce the Plancherel formulas due to (i) Pukanszky, Herb and Wolf (in our notation) (ii) the present author

$$(i) \ x(e) = \frac{2}{\pi^2} \int_0^\infty ds \int_0^1 d\tau s \text{Re} \tanh \pi(s+i\tau) \text{Tr} [T_x^{(\tau,s)}] + \frac{2}{\pi^2} \int_{\frac{1}{2}}^\infty dk \left(k - \frac{1}{2}\right) \text{Tr} [T_x^{k+} + T_x^{k-}] \quad (15a)$$

(Pukanszky-Herb-Wolf)

$$(ii) \ x(e) = \frac{2}{\pi^2} \int_0^\infty ds \int_0^1 d\tau s \text{Re} \tanh \pi \left(s + \frac{i\tau}{2}\right) \text{Tr} [T_x^{(\frac{\tau}{2},s)}] + \frac{2}{\pi^2} \int_{\frac{1}{2}}^\infty dk \left(k - \frac{1}{2}\right) \text{Tr} [T_x^{k+} + T_x^{k-}] \quad (15b)$$

(author)

## II. EVALUATION OF CHARACTER

### A. Elliptic and Hyperbolic elements of the group

The group  $SU(1,1)$  consists of pseudounitary unimodular matrices

$$u = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1$$

and is isomorphic to the real unimodular matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

A particular choice of the isomorphism kernel is

$$\sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

so that  $u = \sigma g \sigma^{-1}$ . Thus

$$\alpha = \frac{1}{2}[(a+d) - i(b-c)], \quad \beta = \frac{1}{2}[(b+c) - i(a-d)]$$

The elements of the group  $SU(1,1)$  or  $(SL(2,R))$  may be divided into three subsets:

(i) elliptic (ii) hyperbolic (iii) parabolic.

We define them as follows. Let

$$\alpha = \alpha_1 + i\alpha_2; \quad \beta = \beta_1 + i\beta_2$$

so that

$$|\alpha|^2 - |\beta|^2 = \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 = 1$$

The elliptic elements are those for which

$$\alpha_2^2 - \beta_1^2 - \beta_2^2 > 0$$

Hence if we set  $\alpha'_2 = \sqrt{\alpha_2^2 - \beta_1^2 - \beta_2^2}$  we have  $\alpha_1^2 + \alpha'^2_2 = 1$ , i.e.  $-1 < \alpha_1 < 1$ . On the other hand the hyperbolic elements are those for which

$$\alpha_2^2 - \beta_1^2 - \beta_2^2 < 0$$

Hence if we write  $\alpha'_2 = \sqrt{\beta_1^2 + \beta_2^2 - \alpha_2^2}$  we have  $\alpha_1^2 - \alpha'^2_2 = 1$  so that  $|\alpha_1| > 1$ . We exclude the parabolic class corresponding to

$$\alpha_2 = \sqrt{\beta_1^2 + \beta_2^2}$$

as this is a submanifold of lower dimensions. In a previous paper[8] we have shown that the elliptic elements can be decomposed as

$$u = \epsilon(\eta)a(\rho)\epsilon(\theta_0)a^{-1}(\rho)\epsilon^{-1}(\eta)$$

$0 < \theta_0 < 2\pi$ ,  $0 \leq \eta \leq 4\pi$ ,  $0 \leq \rho < \infty$ , where

$$\epsilon(\theta_0) = \begin{pmatrix} e^{i\theta_0/2} & 0 \\ 0 & e^{-i\theta_0/2} \end{pmatrix}$$

$$a(\rho) = \begin{pmatrix} \cosh \frac{\rho}{2} & \sinh \frac{\rho}{2} \\ \sinh \frac{\rho}{2} & \cosh \frac{\rho}{2} \end{pmatrix}$$

Thus

$$\alpha = \cos \frac{\theta_0}{2} + i \sin \frac{\theta_0}{2} \cosh \rho$$

$$\beta = -ie^{i\eta} \sin \frac{\theta_0}{2} \sinh \rho$$

The corresponding invariant measure is given by

$$d\mu(u) = \sin^2 \frac{\theta_0}{2} \frac{d\theta_0}{2} \sinh \rho \, d\rho \, d\eta \quad (16a)$$

On the other hand writing the eigenvalues as  $\lambda = \text{sgn} \lambda e^{\pm \text{sgn} \lambda \frac{\sigma}{2}}$ , the hyperbolic elements can be parameterized as[8]

$$\alpha = \text{sgn} \lambda \cosh \frac{\sigma}{2} + i \sinh \frac{\sigma}{2} \sinh \rho \quad (16b)$$

$$\beta = -ie^{-i\theta} \sinh \frac{\sigma}{2} \cosh \rho$$

$0 \leq \theta \leq 4\pi$ ,  $0 \leq \sigma < \infty$ ,  $0 \leq \rho < \infty$ . The invariant measure for the hyperbolic case is given by

$$d\mu(u) = \sinh^2 \frac{\sigma}{2} \frac{d\sigma}{2} \cosh \rho \, d\rho \, d\theta \quad (16c)$$

### B. The character of the principal series of representation

The principal series is realized in the Hilbert space of functions defined on the unit circle  $|z| = 1$ . The finite element of the group is given by,

$$T_u^{(\epsilon, s)} f(z) = (\beta z + \bar{\alpha})^{-k+\epsilon} (\bar{\beta} \bar{z} + \alpha)^{-k-\epsilon} f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right) \quad (17)$$

with  $k = \frac{1}{2} + is$ ,  $0 \leq \epsilon < 1$ . These representations are unitary with respect to the scalar product

$$(f, g) = \int \overline{f(z)} g(z) d\theta, \quad z = e^{i\theta} \quad (18)$$

It will be shown presently that the representation  $\epsilon$  and  $1 - \epsilon$  are unitarily equivalent. Thus the representations  $0 \leq \epsilon \leq \frac{1}{2}$  and  $\frac{1}{2} \leq \epsilon < 1$  are equivalent.

We now construct the operator of the group ring

$$T_x^{(\epsilon, s)} = \int d\mu(u) x(u) T_u^{(\epsilon, s)} \quad (19)$$

where  $x(u)$  is an arbitrary test function on the group which vanishes outside a bounded set. If we define

$$x^\dagger(u) = \overline{x(u^{-1})}$$

we have from Eq. (2),

$$x_1^\dagger x_2(e) = \int \overline{x_1(u^{-1})} x_2(u^{-1}) d\mu(u) = \int \overline{x_1(u)} x_2(u) d\mu(u) \quad (20)$$

The operator of the group ring is now given by

$$T_x^{(\epsilon, s)} f(z) = \int d\mu(u) x(u) (\beta z + \bar{\alpha})^{-k+\epsilon} (\bar{\beta} \bar{z} + \alpha)^{-k-\epsilon} f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right) \quad (21)$$

Setting  $z = -ie^{i\theta}$  and performing the left translation,

$$u \rightarrow \underline{\theta}^{-1} u, \quad \underline{\theta} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \quad (22)$$

we have

$$T_x f(-ie^{i\theta}) = \int d\mu(u) x(\theta^{-1} u) e^{i\theta\epsilon} (-i\beta + \bar{\alpha})^{-k+\epsilon} (i\bar{\beta} + \alpha)^{-k-\epsilon} f\left(\frac{-i\alpha + \bar{\beta}}{-i\beta + \bar{\alpha}}\right) \quad (23)$$

We now perform the Iwasawa decomposition of  $SL(2, R)$

$$g = k\theta_1, \quad k = \begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{pmatrix}, \quad k_{11}k_{22} = 1, \quad \theta_1 = \begin{pmatrix} \cos \frac{\theta_1}{2} & -\sin \frac{\theta_1}{2} \\ \sin \frac{\theta_1}{2} & \cos \frac{\theta_1}{2} \end{pmatrix} \quad (24)$$

so that

$$u = \underline{k} \underline{\theta}_1, \quad \underline{k} = \sigma k \sigma^{-1} \text{ etc.}$$

We now write

$$(i\bar{\beta} + \alpha) = (k_{22} + i0) e^{i\theta_1/2} \quad (25)$$

$$(-i\beta + \bar{\alpha}) = (k_{22} - i0) e^{-i\theta_1/2} \quad (26)$$

and set

$$g(\theta) = f(-ie^{i\theta}) \quad (27)$$

Then we have

$$T_{x,\eta}^{(\epsilon,s)} g(\theta) = \int_0^{2\pi} K_\eta(\theta, \theta_1) g(\theta_1) d\theta_1 \quad (28)$$

with

$$K_\eta(\theta, \theta_1) = \frac{1}{4} \int x(\underline{\theta}^{-1} \underline{k} \underline{\theta}_1) e^{i(\theta - \theta_1)\epsilon} |k_{22}|^{-2k} \times \cos 2\pi\eta\epsilon d\mu_l(k) \quad (29)$$

In obtaining the integral kernel  $K_\eta(\theta, \theta_1)$  of the group ring we have written

$$\begin{aligned} (k_{22} \pm i0) &= |k_{22}| e^{\pm i\pi\eta} \\ \eta &= 0, \text{ for } k_{22} > 0 \\ \eta &= 1, \text{ for } k_{22} < 0 \end{aligned} \quad (30)$$

and following the traditional procedure of classical analysis we have defined the value of the function on the real axis as,

$$f(x) = \frac{1}{2} [f(x + i0) + f(x - i0)] \quad (31)$$

Since the kernel is of the Hilbert-Schmidt type

$$\text{Tr}(T_{x,\eta}^{(\epsilon,s)}) = \int_0^{2\pi} K_\eta(\theta, \theta) d\theta$$

which can be written in the form,

$$\text{Tr}(T_{x,\eta}^{(\epsilon,s)}) = \frac{1}{4} \int_\Theta d\theta \int d\mu_l(k) |k_{22}|^{-2k} x(\underline{\theta}^{-1} \underline{k} \underline{\theta}) \cos 2\pi\epsilon\eta \quad (32)$$

Before proceeding any further we note that  $\underline{\theta}^{-1} \underline{k} \underline{\theta}$  represents a hyperbolic element of  $SU(1, 1)$ :

$$u = \underline{\theta}^{-1} \underline{k} \underline{\theta} \quad (33)$$

Calculating the trace of the  $2 \times 2$  matrix

$$2\alpha_1 = k_{22} + \frac{1}{k_{22}}$$

The above equation for the elliptic element yields

$$k_{22}^2 - 2k_{22} \cos \frac{\theta_0}{2} + 1 = 0$$

which has no real solution. Thus the elliptic elements do not contribute to the character of the principal series of representations.

Following ref. 8 we may now show that every hyperbolic element of  $SU(1, 1)$  can be represented as

$$u = \underline{\theta}^{-1} \underline{k} \underline{\theta}$$

Here  $k_{11} = \lambda^{-1}$ ,  $k_{22} = \lambda$  are the eigenvalues of the matrix  $u$  taken in any order. It follows that for a given choice of  $\lambda$ , the parameters  $\theta$  and  $k_{12}$  are uniquely determined. We note that there are exactly two representations of the matrix  $g = \sigma^{-1} u \sigma \in SL(2, R)$ ,  $\sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$  by means of the formula (33) corresponding to two distinct possibilities

$$\begin{aligned} |k_{11}| &= |\lambda|^{-1} = e^{\sigma/2} & |k_{22}| &= |\lambda| = e^{-\sigma/2} \\ |k_{11}| &= |\lambda|^{-1} = e^{-\sigma/2} & |k_{22}| &= |\lambda| = e^{\sigma/2} \end{aligned}$$

Following Gelfand and Naimark[1] (in  $SL(2, C)$ ) let us now remove from  $K$  the elements with  $k_{11} = k_{22} = 1$ . This operation cuts the group  $K$  into two connected disjoint components. In view of this partition the integral in (32) is represented in the form of the sum of two integrals

$$\text{Tr}(T_{x,\eta}^{(\epsilon,s)}) = \frac{1}{4} \int_{\Theta} d\theta \int_{K_1} d\mu_l(k) |k_{22}|^{-2k} x(\underline{\theta}^{-1} \underline{k} \underline{\theta}) \cos 2\pi\epsilon\eta + \frac{1}{4} \int_{\Theta} d\theta \int_{K_2} d\mu_l(k) |k_{22}|^{-2k} x(\underline{\theta}^{-1} \underline{k} \underline{\theta}) \cos 2\pi\epsilon\eta \quad (34)$$

As  $\theta$  runs over the subgroup  $\Theta$  and  $k$  runs over the components  $K_1$  or  $K_2$  the matrix  $u = \underline{\theta}^{-1} \underline{k} \underline{\theta}$  runs over the hyperbolic elements of  $SU(1, 1)$  (or equivalently  $SL(2, R)$ ). Following ref. 8 it can now be proved that in  $K_1$  or  $K_2$ ,

$$d\mu_l(k) d\theta = \frac{4|k_{22}|d\mu(u)}{|k_{11} - k_{22}|} \quad (35)$$

so that

$$\text{Tr}(T_x^{(\epsilon,s)}) = \sum_{\eta=0,1} \int x(u) \pi_{\eta}^{(\epsilon,s)}(u) d\mu(u) \quad (36)$$

where

$$\pi_{\eta}^{(\epsilon,s)}(u) = \frac{\cos s\sigma}{\sinh \frac{\sigma}{2}} \cos 2\pi\epsilon\eta \quad (37)$$

### C. The positive discrete series $D_k^+$

The finite element of the group for the representation is given by,

$$T_u^{k+} f(z) = (\beta z + \bar{\alpha})^{-2k} f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right) \quad (38)$$

where  $f(z)$  is an analytic function regular within the unit disc. This representation is unitary with respect to the scalar product

$$(f, g) = \int_{|z|<1} \overline{f(z)} g(z) d\lambda(z) \quad (39)$$

where

$$d\lambda(z) = \frac{2k-1}{\pi} (1 - |z|^2)^{2k-2} \quad (40)$$

The integral converges in the usual sense for  $k > \frac{1}{2}$ . For  $0 < k < \frac{1}{2}$  the integral is to be understood in the sense of its regularization (analytic continuation). Thus

$$(f, g) = \frac{2k-1}{2\pi(1-e^{4\pi ik})} \int_{\Sigma} dt(1-t)^{2k-2} \times \int \overline{f(z)} g(z) d\theta \quad (41a)$$

$$z = \sqrt{t}e^{i\theta} \quad (41b)$$

where  $\Sigma$  is a contour (in the  $t$  plane) that starts from the origin along the positive real axis, encircles the point  $+1$  counter-clockwise and returns to the origin along the positive real axis.

The principal vector[9, 10] in this Hilbert space is given by[8],

$$e_z(z_1) = (1 - \bar{z}z_1)^{-2k} \quad (42)$$

so that

$$f(z) = (e_z, f) = \int_{|z_1|<1} (1 - z\bar{z}_1)^{-2k} f(z_1) d\lambda(z_1) \quad (43)$$

The action of the group ring

$$T_x^{k+} = \int x(u) T_u^{k+} d\mu(u) \quad (44)$$

where  $d\mu(u)$  is the invariant measure on  $SU(1, 1)$  is given by,

$$T_x^{k+} f(z) = \int x(u) (\beta z + \bar{\alpha})^{-2k} \times f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right) d\mu(u) \quad (45)$$

Now we use Eq. (43) to write

$$f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right) = \int_{|z_1|<1} \left[1 - \frac{(\alpha z + \bar{\beta})\bar{z}_1}{(\beta z + \bar{\alpha})}\right]^{-2k} f(z_1) d\lambda(z_1) \quad (46)$$

This immediately yields

$$T_x^{k+} f(z) = \int_{|z_1|<1} K(z, z_1) f(z_1) d\lambda(z_1) \quad (47)$$

where

$$K(z, z_1) = \int (\beta z + \bar{\alpha})^{-2k} \left[1 - \frac{(\alpha z + \bar{\beta})\bar{z}_1}{(\beta z + \bar{\alpha})}\right]^{-2k} x(u) d\mu(u) \quad (48)$$

Since the kernel is again of the Hilbert Schmidt type we have

$$\text{Tr}(T_x^{k+}) = \int_{|z|<1} K(z, z) d\lambda(z) \quad (49)$$

which can be written in the form,

$$\text{Tr}(T_x^{k+}) = \int d\mu(u) x(u) \pi^{k+}(u) \quad (50)$$

where

$$\pi^{k+}(u) = \int_{|z|<1} d\lambda(z) [(\beta z + \bar{\alpha}) - (\alpha z + \bar{\beta})\bar{z}]^{-2k} \quad (51)$$

We now substitute

$$z = \tanh \frac{\tau}{2} e^{i\phi}, \quad 0 \leq \tau < \infty; \quad 0 \leq \phi \leq 2\pi \quad (52)$$



Thus, as shown in I, for the elliptic elements (see Eq. (13)).

$$\pi^{k+} = \frac{2k-1}{4\pi} \int_{\tau=0}^{\infty} \int_{\phi=0}^{2\pi} \left[ \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} \hat{n} \cdot \hat{r} \right]^{-2k} \sinh \tau d\phi d\tau \quad (53)$$

where  $\hat{n}$  and  $\hat{r}$  are unit time-like vectors,

$$\begin{aligned} \hat{n} &= (\cosh \rho, -\sinh \rho \sin \eta, \sinh \rho \cos \eta) \\ \hat{r} &= (\cosh \tau, \sinh \tau \sin \phi, \sinh \tau \cos \phi) \end{aligned} \quad (54)$$

and  $\hat{n} \cdot \hat{r}$  is the Lorentz invariant form,

$$\hat{n} \cdot \hat{r} = \hat{n}_3 \hat{r}_3 - \hat{n}_2 \hat{r}_2 - \hat{n}_1 \hat{r}_1$$

We now perform a Lorentz transformation such that the time axis coincides with the time like vector  $\hat{n}$ . Thus

$$\hat{n} \cdot \hat{r} = \cosh \tau$$

and we have

$$\pi^{k+} = \frac{2k-1}{2} \int_0^{\infty} d\tau \sinh \tau \left[ \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} \cosh \tau \right]^{-2k} \quad (55)$$

The integration is quite elementary and we have[11]

$$\pi^{k+}(u) = \frac{e^{\frac{i\theta_0}{2}(2k-1)}}{e^{-\frac{i\theta_0}{2}} - e^{\frac{i\theta_0}{2}}} \quad (56)$$

For the hyperbolic elements we substitute Eq. (16b) and perform a Lorentz transformation ( $\hat{n}$  is now a space-like vector) such that the first space axis coincides with  $\hat{n}$ . Thus

$$\pi^{k+}(u) = \frac{2k-1}{4\pi} \int_0^{\infty} d\tau \sinh \tau \int_0^{2\pi} d\phi \left[ \operatorname{sgn} \lambda \cosh \frac{\sigma}{2} - i \sinh \frac{\sigma}{2} \sinh \tau \cos \phi \right]^{-2k} \quad (57)$$

Evaluation of this integral can be carried out as in ref. 8 and using the definition (31) we have

$$\begin{aligned} \pi^{k+}(u) &= \frac{e^{-\frac{\sigma}{2}(2k-1)}}{e^{\frac{\sigma}{2}} - e^{-\frac{\sigma}{2}}} \cos 2\pi \epsilon' \eta, \quad \eta = 0, 1 \\ k &= n+1 - \epsilon' \quad \text{for } 0 \leq \epsilon' \leq \frac{1}{2} \\ &= n+2 - \epsilon' \quad \frac{1}{2} \leq \epsilon' < 1 \end{aligned} \quad (58)$$

#### D. Negative discrete series $D_k^-$

The finite element of the group in this case is taken to be

$$T_u^{k-} f(z) = (\alpha + \bar{\beta} z)^{-2k} f\left(\frac{\beta + \bar{\alpha} z}{\alpha + \bar{\beta} z}\right) \quad (59)$$

where  $f(z)$  is analytic within the unit disc. The scalar product with respect which  $T_u^{k-}$  is unitary is given by

$$\begin{aligned} (f, g) &= \int \overline{f(z)} g(z) d\lambda(z) \\ d\lambda(z) &= \frac{2k-1}{\pi} (1 - |z|^2)^{2k-2} \end{aligned} \quad (60)$$

The principal vector is as before

$$e_z(z_1) = (1 - \bar{z} z_1)^{-2k}$$

so that

$$f(z) = \int_{|z_1| < 1} (1 - z \bar{z}_1)^{-2k} f(z_1) d\lambda(z_1) \quad (61)$$

Proceeding in the same way as before

$$\pi^{k-}(u) = \int_{|z|<1} [(\alpha + \bar{\beta}z) - (\beta + \bar{\alpha}z)\bar{z}]^{-2k} d\lambda(z) \quad (62)$$

Setting  $z = \tanh \frac{\tau}{2} e^{i\phi}$  we have as before, for the elliptic elements

$$\pi^{k-}(u) = \frac{2k-1}{2} \int d\tau \sinh \tau \left[ \cos \frac{\theta_0}{2} + i \sin \frac{\theta_0}{2} \cosh \tau \right]^{-2k} = \frac{e^{-\frac{i\theta_0}{2}(2k-1)}}{e^{\frac{i\theta_0}{2}} - e^{-\frac{i\theta_0}{2}}} \quad (63)$$

For the hyperbolic elements a parallel calculation gives

$$\pi^{k-}(u) = \frac{e^{-\frac{\sigma}{2}(2k-1)} \cos 2\pi\epsilon'\eta}{[e^{\frac{\sigma}{2}} - e^{-\frac{\sigma}{2}}]}, \quad \eta = 0, 1 \quad (64)$$

and  $\epsilon'$  is defined in the same way as before.

### III. THE PROBLEM OF INVERSION AND THE PLANCHEREL FORMULA

Let us start from,

$$\text{Tr}(T_x^{(\epsilon,s)}) = \sum_{\eta=0,1} \left[ \int_{\text{elliptic}} x(u) \pi^{(\epsilon,s)}(u) d\mu(u) + \int_{\text{hyperbolic}} x(u) \pi_\eta^{(\epsilon,s)}(u) d\mu(u) \right] \quad (65)$$

For the principal series (as indicated by the index  $s$ ) the first term is zero. For the hyperbolic elements

$$d\mu(u) = \sinh^2 \frac{\sigma}{2} \frac{d\sigma}{2} \cosh \rho d\rho d\theta \quad (66a)$$

$$\pi_\eta^{(\epsilon,s)}(u) = \frac{\cos s\sigma}{\sinh \frac{\sigma}{2}} \cos 2\pi\epsilon\eta \quad (66b)$$

Thus if we define

$$\phi_\eta(t) = \int x(\theta, \rho; \eta, t) \cosh \rho d\rho d\theta$$

we have  $\frac{\sigma}{2} = t, \quad \eta = 0, 1$

$$\text{Tr}(T_x^{(\epsilon,s)}) = \int_0^\infty [\phi_0(t) + \cos 2\pi\epsilon\phi_1(t)] \sinh t \cos 2st dt \quad (67)$$

Now we shall divide  $\epsilon$  into two regions,  $0 \leq \epsilon \leq \frac{1}{2}$ , and  $\frac{1}{2} < \epsilon < 1$ . Thus

$$\text{Tr}(T_x^{(\epsilon,s)}) = \theta(2\epsilon)\theta(1-2\epsilon)\text{Tr}(T_x^{(\epsilon,s)}) + \theta(2\epsilon-1)\theta(2-2\epsilon)\text{Tr}(T_x^{(\epsilon,s)})$$

These two regions yield two sets of representations which are unitarily equivalent.

We now have

$$\text{Tr}(T_x^{k+} + T_x^{k-}) = \int [\phi_0(t) + \cos 2\pi\epsilon'\phi_1(t)] \times \sinh t dt e^{-\nu t} - \int_0^\pi d\theta \sin^2 \theta \int \int x(\eta, \rho; \theta) \frac{\sin \nu\theta}{\sin \theta} \times \sinh \rho d\rho d\eta$$

where  $\nu = \nu(\epsilon') = 2k-1$

We now set

$$\int \int x(\eta, \rho; \theta) \sinh \rho d\rho d\eta = F(\theta) \quad (68)$$

so that

$$\int_0^\pi F(\theta) \sin \theta \sin \nu\theta d\theta = \int_0^\infty [\phi_0(t) + \cos 2\pi\epsilon'\phi_1(t)] \sinh t e^{-\nu t} dt - \text{Tr}(T_x^{k+} + T_x^{k-}) \quad (69)$$

Applying the inversion formula for the Fourier cosine transform

$$[\phi_0(t) + \cos 2\pi\epsilon'\phi_1(t)] \sinh t = \frac{4}{\pi} \int_0^\infty ds \cos 2st [\theta(1-2\epsilon') \text{Tr}(T_x^{(\epsilon',s)}) \theta(2\epsilon') + \theta(2\epsilon'-1) \theta(2-2\epsilon') \text{Tr}(T_x^{(\epsilon',s)})] \quad (70)$$

we obtain

$$\begin{aligned} \int F(\theta) \sin \theta \sin \nu \theta d\theta &= \frac{4}{\pi} \int ds \int dt \text{Tr}(T_x^{(\epsilon',s)}) \theta(1-2\epsilon') e^{-\nu t} \cos 2st \theta(2\epsilon') \\ &\quad + \frac{4}{\pi} \int ds \int dt \text{Tr}(T_x^{(1-\epsilon',s)}) \theta(2-2\epsilon') \theta(2\epsilon'-1) e^{-\nu t} \cos 2st - \text{Tr}(T_x^{k+} + T_x^{k-}) \end{aligned} \quad (71)$$

The above formula implies that in the Fourier sine transform of the function  $F(\theta) \sin \theta$ ,

$$\begin{aligned} F(\theta) \sin \theta &= \frac{2}{\pi} \int_0^\infty c(\nu) \sin \nu \theta d\nu \\ c(\nu) &= \frac{4}{\pi} \int ds \int dt \text{Tr}(T_x^{(\epsilon',s)}) e^{-\nu t} \theta(1-2\epsilon') \cos 2st \theta(2\epsilon') \\ &\quad + \frac{4}{\pi} \int ds \int dt \text{Tr}(T_x^{(1-\epsilon',s)}) e^{-\nu t} \theta(2-2\epsilon') \theta(2\epsilon'-1) \cos 2st - \text{Tr}(T_x^{k+} + T_x^{k-}) \end{aligned} \quad (72)$$

Thus we have

$$\begin{aligned} F(\theta) \sin \theta &= \frac{8}{\pi^2} \int ds \int dt \theta(1-2\epsilon') \theta(2\epsilon') \text{Tr}(T_x^{(\epsilon',s)}) \int e^{-\nu t} \cos 2st \sin \nu \theta d\nu \\ &\quad + \frac{8}{\pi^2} \int ds \int dt \theta(2-2\epsilon') \theta(2\epsilon'-1) \text{Tr}(T_x^{(1-\epsilon',s)}) \cos 2st \int e^{-\nu t} \sin \nu \theta d\nu \\ &\quad - \int_0^\infty \text{Tr}(T_x^{k+} + T_x^{k-}) \sin \nu \theta d\nu \end{aligned}$$

In the first integral,

$$k = n + 1 - \epsilon' \quad \nu = 2k - 1 = 2n + 1 - \tau \quad d\nu = -d\tau$$

Similarly in the second

$$k = n + 2 - \epsilon' \quad \nu = 2n + 1 + \tau \quad \tau = 2 - 2\epsilon'; \quad d\nu = d\tau$$

Thus we have

$$\begin{aligned} F(\theta) \sin \theta &= \frac{8}{\pi^2} \int_0^\infty ds \int_0^1 d\tau \text{Tr}(T_x^{(\frac{\tau}{2},s)}) \int_0^\infty dt \cos 2st \\ &\quad \times \sum_{n=0}^\infty [e^{-(2n+1-\tau)t} \sin(2n+1-\tau)\theta + e^{-(2n+1+\tau)t} \sin(2n+1+\tau)\theta] \\ &\quad - \frac{2}{\pi} \int \text{Tr}(T_x^{k+} + T_x^{k-}) \sin \nu \theta d\nu \end{aligned} \quad (73)$$

The summation can be easily carried out recalling that

$$\sum e^{-(2n+1\mp\tau)t} \sin(2n+1\mp\tau)\theta = \text{Im} e^{-t(1\mp\tau)+i(1\mp\tau)\theta} \times \sum e^{-2nt+2in\theta} \quad (74)$$

and using the standard summation formula for the geometric series. Thus

$$\begin{aligned} F(\theta) \sin \theta &= \frac{8}{\pi^2} \int_0^\infty ds \int_0^1 d\tau \text{Tr}(T_x^{(\frac{\tau}{2},s)}) \times \left[ \sin(1+\tau)\theta \int_0^\infty dt \frac{\cos 2st \cosh(1-\tau)t}{\cosh 2t - \cos 2\theta} \right. \\ &\quad \left. + \sin(1-\tau)\theta \int_0^\infty dt \frac{\cos 2st \cosh(1+\tau)t}{\cosh 2t - \cos 2\theta} \right] - \frac{2}{\pi} \int_0^\infty d\nu \sin \nu \theta \text{Tr}(T_x^{k+} + T_x^{k-}) \end{aligned} \quad (75)$$

The integrals can be written in the form,

$$\frac{1}{2} \int_{-\infty}^\infty e^{2ist} \frac{\cosh(1\mp\tau)t dt}{\cosh 2t - \cos 2\theta}$$

which can be evaluated by the sum of the residues at the poles in the upper half-plane located at

$$t = i(\theta + n\pi), \quad n = 0, 1, 2, \dots$$

and at

$$t = i(n\pi - \theta), \quad n = 1, 2, 3, \dots$$

It can be easily ascertained that

$$\sin(1 + \tau)\theta \int_0^\infty \frac{\cos 2st \cosh(1 - \tau)t}{\cosh 2t - \cos 2\theta} dt + \sin(1 - \tau)\theta \int_0^\infty \frac{\cos 2st \cosh(1 + \tau)t}{\cosh 2t - \cos 2\theta} dt = \frac{\pi}{2} \operatorname{Re} \frac{\cosh \left[ \pi \left( s + \frac{i\tau}{2} \right) - 2s\theta \right]}{\cosh \pi \left( s + \frac{i\tau}{2} \right)}$$

We therefore obtain

$$F(\theta) \sin \theta = \frac{4}{\pi} \int_0^\infty ds \int_0^1 d\tau \operatorname{Tr}(T_x^{(\frac{\tau}{2}, s)}) \operatorname{Re} \frac{\cosh \left[ \pi \left( s + \frac{i\tau}{2} \right) - 2s\theta \right]}{\cosh \pi \left( s + \frac{i\tau}{2} \right)} - \frac{4}{\pi} \int dk \sin(2k - 1)\theta \operatorname{Tr}[T_x^{k+} + T_x^{k-}] \quad (76)$$

(b) It now remains to relate  $x(e)$  with  $F(\theta) \sin \theta$ . For this we equate two different calculations for the residue at the pole at  $\lambda = -\frac{3}{2}$  of the generalized function

$$(x_3^2 - x_2^2 - x_1^2)_+^\lambda$$

as an analytic function of  $\lambda$ . Let us consider

$$I(\lambda) = \int_{x_0 > 0} (x_3^2 - x_2^2 - x_1^2)_+^\lambda x(u) d\mu(u) \quad (77)$$

where  $d\mu(u)$  stands for the invariant measure for the elliptic elements (as given by Eq. (14)) and

$$x_3 = \sin \theta \cosh \rho \quad x_2 = \sin \theta \sinh \rho \sin \eta$$

$$x_1 = \sin \theta \sinh \rho \cos \eta; \quad 0 \leq \eta \leq 4\pi, \quad 0 \leq \rho < \infty; \quad 0 \leq \theta \leq \pi$$

where  $\theta = \frac{\theta_0}{2}$ ,  $\rho, \eta$  are defined by Eq. (13) and  $x_0 = \cos \theta$ .

It now immediately follows

$$d\mu(u) = \frac{dx_1 dx_2 dx_3}{|x_0|} \quad (78)$$

Thus

$$I(\lambda) = \int_{x_0 > 0} (x_3^2 - x_2^2 - x_1^2)_+^\lambda \phi(x) dx_1 dx_2 dx_3 \quad (79)$$

$$\phi(x) = \frac{x(u)}{|x_0|}$$

Setting

$$x_1^2 + x_2^2 = v \quad x_3^2 = u$$

$$v = ut$$

we have

$$I(\lambda) = -\frac{1}{4} \int du u^{\lambda + \frac{1}{2}} \int_0^1 (1 - t)^\lambda \psi(u, tu) dt \quad (80)$$

where

$$\psi(u, tu) = \int_0^{4\pi} \phi(\sqrt{ut} \cos \eta, \sqrt{ut} \sin \eta, \sqrt{u}) d\eta \quad (81)$$

Since  $\phi(x)$  vanishes outside a bounded set the integral  $I(\lambda)$  converges in the usual sense for  $\text{Re}\lambda > -1$ . For  $\text{Re}\lambda < -1$  it is to be understood in the sense of its regularization (analytic continuation) :

$$I(\lambda) = \frac{1}{4} \frac{1}{(e^{2\pi i\lambda} - 1)(e^{2\pi i\lambda} + 1)} \int_{-\infty}^{0+} du u^{\lambda+\frac{1}{2}} \int_0^{1+} (1-t)^\lambda \psi(u, tu) dt \quad (82)$$

We now define

$$\Phi(\lambda, u) = -\frac{1}{4} \frac{1}{(e^{2\pi i\lambda} - 1)} \int_0^{1+} (1-t)^\lambda \psi(u, tu) dt \quad (83)$$

Just as the generalized function  $(x_+^\lambda, \phi)$   $\Phi(\lambda, u)$  is regular for all  $\lambda$  except at

$$\lambda = -1, -2, -3, \dots$$

where it has simple poles.

On the other hand at regular points of  $\Phi(\lambda, u)$  the integral

$$I(\lambda) = -\frac{1}{(e^{2\pi i\lambda} + 1)} \int_{-\infty}^{0+} du u^{\lambda+\frac{1}{2}} \Phi(\lambda, u) \quad (84)$$

also have poles at

$$\lambda = -\frac{3}{2}, -\frac{5}{2}, \dots$$

which are once again simple poles. The analytic function  $I(\lambda)$  can be written as

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{\frac{1}{n!} \left[ \frac{\partial^n}{\partial u^n} \Phi(\lambda, u) \right]_{u=0}}{\lambda + \frac{3}{2} + n} + E(\lambda) \quad (85)$$

where  $E(\lambda)$  is an entire function. Thus

$$\text{Res}[I(\lambda)]_{\lambda=-\frac{3}{2}} = \Phi\left(-\frac{3}{2}, 0\right) = -\frac{1}{2} \psi(0, 0)$$

Now

$$\psi(0, 0) = \int_0^{4\pi} \phi(0, 0, 0) d\eta = 4\pi x(e)$$

Thus

$$\text{Res}[I(\lambda)]_{\lambda=-\frac{3}{2}} = -2\pi x(e) \quad (86)$$

We shall calculate the same thing in another way by writing

$$I(\lambda) = \int_0^{\frac{\pi}{2}} \sin^{2\lambda+1} \theta [F(\theta) \sin \theta]$$

We now write  $\sin \theta = \theta[1 - u(\theta)]$  where

$$u(0) = \left[ \frac{du}{d\theta} \right]_{\theta=0} = 0$$

$$\text{Thus } I(\lambda) = \int_0^{\frac{\pi}{2}} \theta^{2\lambda+1} G(\theta) d\theta \quad (87a)$$

$$\text{where } G(\theta) = F(\theta) \sin \theta [1 - u(\theta)]^{2\lambda+1} \quad (87b)$$

Since  $G(\theta)$  has a compact support and is regular at  $\theta = 0$  we have[3] for  $\text{Re}\lambda > -\frac{n}{2} - 1$

$$I(\lambda) = \int_0^{\frac{\pi}{2}} \theta^{2\lambda+1} \left[ G(\theta) - \sum_{r=0}^{n-1} \frac{G^{(r)}(0)\theta^r}{r!} \right] + \sum_r \frac{G^{(r)}(0)}{r!(2\lambda+2+r)} \quad (88)$$

Hence

$$\text{Res}[I(\lambda)]_{\lambda=-\frac{3}{2}} = \frac{1}{2}G'(0) = \frac{1}{2} \left\{ \frac{d}{d\theta} [\sin \theta F(\theta)] \right\}_{\theta=0} \quad (89)$$

Equating Eqs. (86) and (89) we have

$$x(e) = -\frac{1}{4\pi} \left\{ \frac{d}{d\theta} [\sin \theta F(\theta)] \right\}_{\theta=0} \quad (90)$$

Combining Eqs. (76) and (90) we immediately obtain

$$x(e) = \frac{2}{\pi^2} \int_0^\infty ds \int_0^1 d\tau \text{Tr}(T_x^{(\frac{\tau}{2}, s)}) \times s \text{Re} \tanh \pi \left( s + \frac{i\tau}{2} \right) + \frac{2}{\pi^2} \int_{\frac{1}{2}}^\infty dk \left( k - \frac{1}{2} \right) \text{Tr}(T_x^{k+} + T_x^{k-}) \quad (91)$$

Replacing  $x(e)$  by  $x_1^\dagger x_2(e)$  and using Eq. (20) we have

$$\begin{aligned} \int \overline{x_1(u)} x_2(u) d\mu(u) &= \frac{2}{\pi^2} \int_0^\infty ds \int_0^1 d\tau \text{Tr}[T_{x_1}^{(\frac{\tau}{2}, s)^\dagger} T_{x_2}^{(\frac{\tau}{2}, s)}] s \text{Re} \tanh \pi \left( s + \frac{i\tau}{2} \right) \\ &\quad + \frac{2}{\pi^2} \int_{\frac{1}{2}}^\infty dk \left( k - \frac{1}{2} \right) \text{Tr}(T_{x_1}^{k+^\dagger} T_{x_2}^{k+} + T_{x_1}^{k-^\dagger} T_{x_2}^{k-}) \end{aligned} \quad (92)$$

This is the analogue of the Plancherel formula for the ordinary Fourier transform.

#### IV. ACKNOWLEDGEMENT

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